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ABSTRACT

Ordinary differential equations with greatly different time constants arise in a wide variety of important physical problems. The predictor-corrector method of solution is examined here.

The existence of stiffly stable methods of order as high as eight has been known for some time. In this report, we have examined a class of methods for finding methods of order greater than eight. We find that the choice of

$$\sigma(\xi) = \xi^{k-r} (\xi-c)^r, \; r = 0, 1, 2, \ldots k \text{ and } -1 \leq c < 1$$

leads to the methods of order k as high as eleven for some c and r. The multistep methods are stiffly stable for $k \leq 9$ if $r = 3$, for $k \leq 10$ if $r = 4$, and for $k \leq 11$ if $r = 6$.

We found no twelfth order and higher stiffly stable methods. However, if we select

$$(P_r) \quad \sigma(\xi) = \xi^{k-r} (\xi^r - c^r) \text{ or}$$

$$(Q_r) \quad \sigma(\xi) = \xi^{k-r} \sum_{i=0}^{r} \xi^{r-i} c^i$$

where $r = 0, 1, 2, \ldots k$ and $-1 \leq c < 1$.

We obtain stiffly stable methods of P_r type if $k \leq 6$ and of Q_r type if $k \leq 7$.

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1. INTRODUCTION

The mathematical analysis of the problems encountered in reactor calculations, circuit analysis and chemical kinetics often leads to stiff systems of ordinary differential equations, i.e. systems with widely separated time constants. The standard methods for the numerical integration of the ordinary differential equations are not well suited to the stiff differential equations for reasons of stability, the time steps have to be adopted to the fast decaying components while one is interested in the slowly decaying ones.

A number of authors have proposed special algorithms to overcome this difficulty (Lawson [1], Richard, Lanning and Torrey [2], and Liniger and Willoughby [3]). However, there has remained still a strong desire to develop methods of Runge-Kutta or multistep type with improved stability and accuracy properties so that the above difficulty is diminished. The object of this investigation is to develop high order stiffly stable multistep methods. A linear k-step method for the numerical solution of

$$y' = f(t, y),\ y(t_0) = \eta$$

can be written in the form

(1)
$$\alpha_0\, y_{n+1} = \sum_{i=1}^{k} \alpha_i\, y_{n+1-i} + h \sum_{i=0}^{k} \beta_i\, y'_{n+1-i}$$

with $\alpha_0 \neq 0$ and $|\alpha_k| + |\beta_k| \neq 0$ or symbolically,

(2)
$$\rho(E)\, y_{n+1-k} = h\, \sigma(E)\, y'_{n+1-k}$$

where E is the translation operator ($Ey_n = y_{n+1}$) and ρ and σ are polynomials defined by

$$\rho(\xi) = \alpha_0\, \xi^k - \alpha_1\, \xi^{k-1} - \alpha_2\, \xi^{k-2} - \ldots - \alpha_k$$

(3)
$$\sigma(\xi) = \beta_0\, \xi^k + \beta_1\, \xi^{k-1} + \beta_2\, \xi^{k-2} + \ldots + \beta_k$$

Therefore, this formula can only be used if one knows the values of the solution at k successive points. These k values will be assumed to be given. Further, it can be assumed without loss of generality that the polynomials $\rho(\xi)$ and $\sigma(\xi)$ have no common factors since in general case, (1) can be reduced to an equation of lower order.

<u>Definition 1.1</u>. *The formula (1) will be said to be of order $p \geq 0$ if it fulfills the $p + 1$ conditions*

(4)
$$\alpha_0 = \sum_{i=1}^{k} \alpha_i$$

$$\alpha_0 = \sum_{i=1}^{k} (1-i)^s \alpha_i + s \sum_{i=0}^{k} (1-i)^{s-1} \beta_i, \quad s = 1, 2, \ldots p$$

Thus the method is of order p if for any $y \in c^{(p+2)}$ and for some nonzero c_{p+1}.

(5)
$$\alpha_0 y_{n+1} = \sum_{i=1}^{k} \alpha_i y_{n+1-i} + h \sum_{i=0}^{k} \beta_i y'_{n+1-i} + c_{p+1} h^{p+1} y^{(p+1)}_{(t)} + O(h^{p+2})$$

where $y^{(p+1)}$ is the (p+1)-st derivative of y evaluated for some t between t_{n+1-k} and t_{n+1}. The last two terms represent the truncation error.

Multistep methods were first investigated by Dahlquist [4] who defined the following:

<u>Definition 1.2</u>. *A multistep method is called A-stable if all solutions of (1) tend to zero, as $n \to \infty$, when it is applied to the differential equation of the form $y' = \lambda y$ and λ is a (complex) constant with negative real parts.*

He then proved that the order p of an A-stable linear multistep method cannot exceed two. The order two method with the smallest error term is the trapezoidal rule. Widlund [5] has shown the

existence of methods of orders three and four which are stable in the wedge $|\arg(h\lambda) - \pi| < \alpha$ of the negative half plane for any $\alpha < \frac{\pi}{2}$. Norsett [6] has extended these results to the methods of orders five and six. Gear [7,8] has used the conditions which are necessary for stiff differential equations. The requirements are shown in Figure 1.

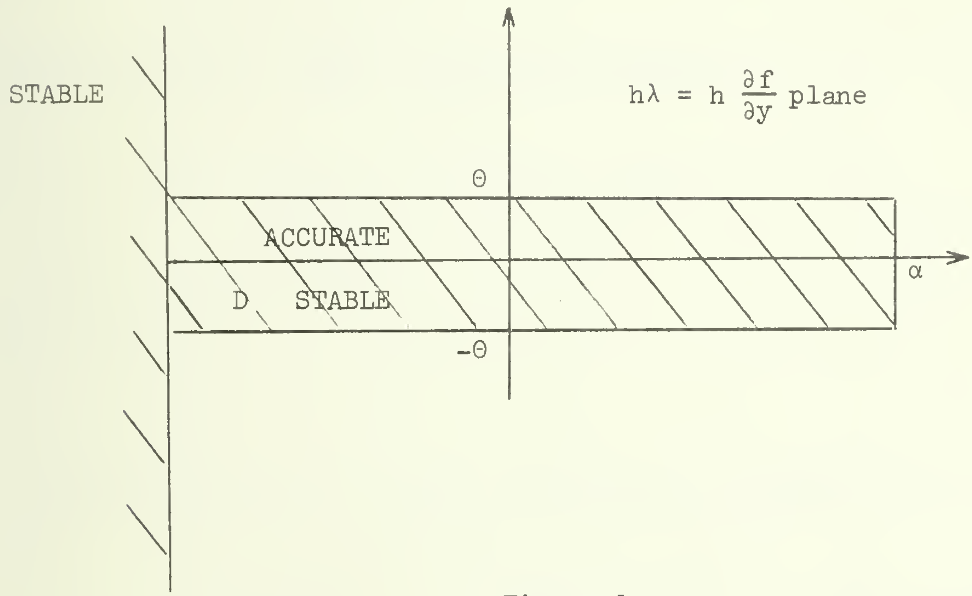

Figure 1

Stability and Accuracy Region

Thus the numerical methods suitable for stiff differential equations depend on the parameters D, Θ, α and on the definition of accuracy. Gear has termed such methods as stiffly stable methods and has obtained methods of order as high as six for suitable parameters D, Θ, and α. Dill [9] has used computer graphic techniques for finding methods of orders seven and eight. We shall show here that the methods of order as high as eleven can be obtained for suitable parameters.

2. HIGH ORDER STIFFLY STABLE METHODS

The discretization error of the multistep method is defined as the difference between the value y_n calculated from (1) and the exact solution $y(t_n)$.

Define

$$\varepsilon_n = y_n - y(t_n)$$

Then the error ε_n obeys the difference equation

(6)
$$(\alpha_0 - h\lambda\beta_0)\varepsilon_{n+1} = \sum_{i=1}^{k} (\alpha_i + h\lambda\beta_i)\varepsilon_{n+1-i} - T_n$$

for the equation $y' = \lambda y$.

Assuming that

$$\varepsilon_i = \xi^i$$

we get from (6) the well known characteristic equation for ξ,

(7)
$$(\alpha_0 - h\lambda\beta_0)\xi^{n+1} - \sum_{i=1}^{k} (\alpha_i + h\lambda\beta_i)\xi^{n+1-i} + T_n(y, t_n, h) = 0$$

where T_n is the truncation error for the exact solution. For convergence, it is necessary to bound the solution of the imhomogeneous difference equation (7) (see, Henrici [10]). It depends on the stability of the corresponding homogeneous equation obtained from (7) with T_n set to zero. This difference equation is stable if and only if all the roots of the polynomial equation

$$(\alpha_0 - h\lambda\beta_0)\xi^{n+1} - \sum_{i=1}^{k} (\alpha_i + h\lambda\beta_i)\xi^{n+1-i} = 0$$

or symbolically

(8) $\rho(\xi) - h\lambda\, \sigma(\xi) = 0$

lie in or on the unit circle and those lying on the unit circle have multiplicity of one. It should be noted that it is an asymptotic condition only and is concerned with the convergence of y_n to $y(t)$ with $h = (t - t_0)/n$ as $n \to \infty$ or $h\lambda \to 0$. If stiff equations are to be integrated with large values of h, then large values of $h\lambda$ must not make (8) unstable. Letting $h\lambda \to \infty$, the roots of (8) approach those of $\sigma(\xi) = 0$. This implies that the polynomial $\sigma(\xi)$ must not have roots outside the unit circle and those roots ξ_i for which $|\xi_i| = 1$ are simple. Further, $\sigma(\xi)$ is of degree k and has no common root with $\rho(\xi)$. The stiff stability can be investigated as follows.

We want $h\lambda$ values such that (8) has roots inside the unit circle or on the unit circle and simple. The region is bounded by the locus of $\rho(\xi)/\sigma(\xi)$ in the $h\lambda$-plane for $\xi = e^{i\theta}$, $\theta \in [0,2\pi]$. This locus can be plotted in $h\lambda$ plane and is of the type as shown in Figure 2. At $h\lambda$ = infinity, the method is stable if $\sigma(\xi)$ is stable, so that any region connected to that point will be stable by continuity argument. If we prescribe k, then for stiffly stable methods the polynomial $\sigma(\xi)$ should be such that its roots lie within the unit circle or on the unit circle and simple. Another restriction on $\sigma(\xi)$ is that it has no common factor with $\rho(\xi)$. Since $\rho(\xi)$ will always have a root at +1, $\sigma(\xi)$ must not have a root at +1. The locus of $\rho(\xi)/\sigma(\xi)$ for $\xi = e^{i\theta}$, $\theta \in [0,2\pi]$ should be as shown in Figure 2.

Selecting Coefficients of $\sigma(\xi)$

We shall consider the stiffly stable methods for which k = p. A k-step stiffly stable method of order p (=k) requires the solution of p + 1 equations in 2k + 1 unknowns (β_0 can be taken as 1). The k + 1 unknowns α_i, the coefficients of the polynomial $\rho(\xi)$ can be determined in terms of k unknowns β_i, the coefficients of the polynomial $\sigma(\xi)$. The expressions for α_i in terms of β_i as obtained on solving the equations (4) are given in Appendix 1.

Besides the above metnioned restrictions on $\sigma(\xi)$, these arbitrary coefficients β_i can be chosen to

1. control numerical stability,

2. minimize computational efforts, and

3. minimize the truncation error.

Gear has chosen $\beta_i = 0$, $i = 1, 2, \ldots k$, i.e. $\sigma(\xi) = \xi^k$ and obtained stiffly stable methods for $k \leq 6$. $k \geq 7$ does not give a stable method. These methods satisfy (1) and (2). Dill has prescribed $\beta_0 = 1$ and $\beta_1 = -.99$, i.e. $\sigma(\xi) = \xi^{k-1}(\xi - 0.99)$ and obtained a seventh order formula. For eighth order formula he has selected $\sigma(\xi) = \xi^{k-2}(\xi^2 - 1.8\xi + .81)$. These formulas also satisfy the criterion (2).

We have investigated a class of methods which will satisfy the essential property (1). Some of our formulas will also satisfy (2) and (3).

<u>Class I</u>

Choose

(9) $$\sigma(\xi) = \xi^{k-r}(\xi - c)^r, \quad r = 0, 1, 2, \ldots k$$

where

$$-1 \leq c < 1.$$

We find

r	0	1	2	3	4	6
order of stable formula	$k \leq 6$	$k \leq 7$	$k \leq 8$	$k \leq 9$	$k \leq 10$	$k \leq 11$

There are no twelfth order and higher stiffly stable methods. Appendix 2 contains some of the special case formulas. The values of the parameters D, c, $\max|\xi|$ (ξ_i being the roots of $\rho(\xi) = 0$) and c_{p+1} (the coefficients of the truncation error) are also given. The section of the locus $\rho(\xi)/\sigma(\xi)$ for seventh to eleventh order formulas for different values of r are shown in Figures 3 to 7, respectively.

Class II

Here we take $\sigma(\xi)$ of the following form

$$(P_r) \qquad \sigma(\xi) = \xi^{k-r} (\xi^r - c^r)$$

$$(Q_r) \qquad \sigma(\xi) = \xi^{k-r} \sum_{i=0}^{k} \xi^{r-i} c^i$$

where $r = 0, 1, 2, \ldots k$ and $-1 \leq c < 1$.

The above P_r and Q_r type formulas for $r = 0$ reduce to the case considered by Gear and are stiffly stable for $k \leq 6$.

By prescribing the values of **c** consistent with the stability requirement, a locus was obtained which indicated the existence of suitable parameters D and Θ for a stiffly stable method. These results are given in the table below.

r	0	1	2	3	4	5	6
P_r	$k \leq 6$	$k \leq 7$	$k \leq 6$	$k \leq 7$	$k \leq 6$	$k \leq 6$	$k \leq 6$
Q_r	$k \leq 6$	$k \leq 7$	$k \leq 7$	$k \leq 6$	$k \leq 6$	$k \leq 6$	$k \leq 6$

$r > 6$ does not indicate the existence of stiffly stable methods. Appendix 3 shows the coefficients for various order formulas of P_r and Q_r type. The values of the parameters c, D, $\max|\xi|$ and c_{p+1} are also tabulated. The section of the locus of $\rho(\xi)/\sigma(\xi)$ for fifth order method is shown in Figure 8.

3. CONCLUSIONS

The stiffly stable methods of order as high as eight are already known. The aim of the present investigation has been to develop methods of order higher than eight.

We have investigated the following classes of methods:

(1) $\sigma(\xi) = \xi^{k-r} (\xi - c)^r$

(2) $\sigma(\xi) = \xi^{k-r} (\xi^r - c^r)$

(3) $\sigma(\xi) = \xi^{k-r} \sum_{i=0}^{r} \xi^{r-i} c^i$

where $r = 0, 1, 2, \ldots k$ and $-1 \leq c < 1$.

We find that the choice (1) leads to the methods of order as high as eleven. The methods are stiffly stable for $k \leq 9$ if $r = 3$, for $k \leq 10$ if $r = 4$, and for $k \leq 11$ if $r = 6$. Further, we found no twelfth order and higher methods. The stiffly stable methods of the types (2) and (3) do not exist for $k \geq 8$. A comparitive study of the methods developed in this report is under investigation.

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APPENDIX I

A. Third Order Formula:

$$\alpha_0 \, y_{n+1} = \sum_{i=0}^{2} \alpha_{i+1} \, y_{n-i} + h \sum_{i=0}^{3} \beta_i \, y'_{n+1-i}$$

(1.1)

$$6 \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 11 & 2 & -1 & 2 \\ 18 & -3 & -6 & 9 \\ -9 & 6 & 3 & -18 \\ 2 & -1 & 2 & 11 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

and

$$T_n = -(3\beta_0 - \beta_1 + \beta_2 - 3\beta_3) \frac{h^4}{12} y_n^{(4)}$$

B. Fourth Order Formula:

$$\alpha_0\, y_{n+1} = \sum_{i=0}^{3} \alpha_{i+1}\, y_{n-i} + h \sum_{i=0}^{4} \beta_i\, y'_{n+1-i}$$

(1.2)

$$12 \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 25 & 3 & -1 & 1 & -3 \\ 48 & -10 & -8 & 6 & -16 \\ -36 & 18 & 0 & -18 & 36 \\ 16 & -6 & 8 & 10 & -48 \\ -3 & 1 & -1 & 3 & 25 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}$$

and

$$T_n = -(12\beta_0 - 3\beta_1 + 2\beta_2 - 3\beta_3 + 12\beta_4)\,\frac{h^5}{60}\, y_n^{(5)}$$

C. Fifth Order Formula:

$$\alpha_0 \, y_{n+1} = \sum_{i=0}^{4} \alpha_{i+1} \, y_{n-i} + h \sum_{i=0}^{5} \beta_i \, y'_{n+1-i}$$

(1.3)

$$60 \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} 137 & 12 & -3 & 2 & -3 & 12 \\ 300 & -65 & -30 & 15 & -20 & 75 \\ -300 & 120 & -20 & -60 & 60 & -200 \\ 200 & -60 & 60 & 20 & -120 & 300 \\ -75 & 20 & -15 & 30 & 65 & -300 \\ 12 & -3 & 2 & -3 & 12 & 137 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix}$$

and

$$T_n = -(10\beta_0 - 2\beta_1 + \beta_2 - \beta_3 + 2\beta_4 - 10\beta_5) \frac{h^6}{60} y_n^{(6)}$$

D. Sixth Order Formula:

$$\alpha_0 \, y_{n+1} = \sum_{i=0}^{5} \alpha_{i+1} \, y_{n-1} + h \sum_{i=0}^{6} \beta_i \, y'_{n+1-i}$$

(1.4)

$$60 \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} = \begin{bmatrix} 147 & 10 & -2 & 1 & -1 & 2 & -10 \\ 360 & -77 & -24 & 9 & -8 & 15 & -72 \\ -450 & 150 & -35 & -45 & 30 & -50 & 225 \\ 400 & -100 & 80 & 0 & -80 & 100 & -400 \\ -225 & 50 & -30 & 45 & 35 & -150 & 450 \\ 72 & -15 & 8 & -9 & 24 & 77 & -360 \\ -10 & 2 & -1 & 1 & -2 & 10 & 147 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix}$$

and

$$T_n = -(60\beta_0 - 10\beta_1 + 4\beta_2 - 3\beta_3 + 4\beta_4 - 10\beta_5 + 60\beta_6) \frac{h^7}{420} y_n^{(7)}$$

E. Seventh Order Formula:

$$\alpha_0 \, y_{n+1} = \sum_{i=0}^{6} \alpha_{i+1} \, y_{n-i} + h \sum_{i=0}^{7} \beta_i \, y'_{n+1-i}$$

(1.5)

$$420 \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{bmatrix} = \begin{bmatrix} 1089 & 60 & -10 & 4 & -3 & 4 & -10 & 60 \\ 2940 & -609 & -140 & 42 & -28 & 35 & -84 & 490 \\ -4410 & 1260 & -329 & -252 & 126 & 140 & 315 & 1764 \\ 4900 & -1050 & 700 & -105 & -420 & 350 & -700 & 3675 \\ -3675 & 700 & -350 & 420 & 105 & -700 & 1050 & -4900 \\ 1764 & -315 & 140 & -126 & 252 & 329 & -1260 & 4410 \\ -490 & 84 & -35 & 28 & -42 & 140 & 609 & -2940 \\ 60 & -10 & 4 & -3 & 4 & -10 & 60 & 1089 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \end{bmatrix}$$

and

$$T_n = -(105\beta_0 - 15\beta_1 + 5\beta_2 - 3\beta_3 + 3\beta_4 - 5\beta_5 + 15\beta_6 - 105\beta_7) \frac{h^8}{840} y_n^{(8)}$$

F. Eighth Order Formula:

$$\alpha_0 y_{n+1} = \sum_{i=0}^{7} \alpha_{i+1} y_{n-i} + h \sum_{i=0}^{8} \beta_i y'_{n+1-i}$$

(1.6)

$$840 \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix} = \begin{bmatrix} 2283 & 105 & -15 & 5 & -3 & 3 & -5 & 15 & -105 \\ 6720 & -1338 & -240 & 60 & -32 & 30 & -48 & 140 & -960 \\ -11760 & 2940 & -798 & -420 & 168 & -140 & 210 & -588 & 3920 \\ 15680 & -2940 & 1680 & -378 & -672 & 420 & -560 & 1470 & -9408 \\ -14700 & 2450 & -1050 & 1050 & 0 & -1050 & 1050 & -2450 & 14700 \\ 9408 & 1470 & 560 & -420 & 672 & 378 & -1680 & 2940 & -15680 \\ -3920 & 588 & -210 & 140 & -168 & 420 & 798 & -2940 & 11760 \\ 960 & -140 & 48 & -30 & 32 & -60 & 240 & 1338 & -6720 \\ -105 & 15 & -5 & 3 & -3 & 5 & -15 & 105 & 2283 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \end{bmatrix}$$

and

$$T_n = -(280\beta_0 - 35\beta_1 + 10\beta_2 - 5\beta_3 + 4\beta_4 - 5\beta_5 + 10\beta_6 - 35\beta_7 + 280\beta_8) \frac{h^9}{2520} y_n^{(9)}$$

G. Ninth Order Formula:

$$\alpha_0\, y_{n+1} = \sum_{i=0}^{8} \alpha_{i+1}\, y_{n-i} + h \sum_{i=0}^{9} \beta_i\, y'_{n+1-i}$$

(1.7)

$$2520 \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \end{bmatrix} = \begin{bmatrix} 7129 & 280 & -35 & 10 & -5 & 4 & -5 & 10 & -35 & 280 \\ 22680 & -4329 & -630 & 135 & -60 & 45 & -54 & 105 & -360 & 2835 \\ -45360 & 10080 & -2754 & -1080 & 360 & -240 & 270 & -504 & 1680 & -12960 \\ 70560 & -11760 & 5880 & -1554 & -1680 & 840 & -840 & 1470 & -4704 & 35280 \\ -79380 & 11760 & -4410 & 3780 & -504 & -2520 & 1890 & -2940 & 8820 & -63504 \\ 63504 & -8820 & 2940 & -1890 & 2520 & 504 & -3780 & 4410 & -11760 & 79380 \\ -35280 & 4704 & -1470 & 840 & -840 & 1680 & 1554 & -5880 & 11760 & -70560 \\ 12960 & -1680 & 504 & -270 & 240 & -360 & 1080 & 2754 & -10080 & 45360 \\ -2835 & 360 & -105 & 54 & -45 & 60 & -135 & 630 & 4329 & -22680 \\ 280 & -35 & 10 & -5 & 4 & -5 & 10 & -35 & 280 & 7129 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \end{bmatrix}$$

and

$$T_n = -(252\beta_0 - 28\beta_1 + 7\beta_2 - 3\beta_3 + 2\beta_4 - 2\beta_5 + 3\beta_6 - 7\beta_7 + 28\beta_8 - 252\beta_9)\, \frac{h^{10}}{2520}\, y_n^{(10)}$$

H. Tenth Order Formula:

$$\alpha_0 y_{n+1} = \sum_{i=0}^{9} \alpha_{i+1} y_{n-i} + h \sum_{i=0}^{10} \beta_i y'_{n+1-i}$$

(1.8)

$$2520 \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \end{bmatrix} = \begin{bmatrix} 7381 & 252 & -28 & 7 & -3 & 2 & -2 & 3 & -7 & 28 & -252 \\ 25200 & -4609 & -560 & 105 & -40 & 25 & -24 & 35 & -80 & 315 & -2800 \\ -56700 & 11340 & -3096 & -945 & 270 & -150 & 135 & -189 & 420 & -1620 & 14175 \\ 100800 & -15120 & 6720 & -1914 & -1440 & 600 & -480 & 630 & -1344 & 5040 & -43200 \\ -132300 & 17640 & -5880 & 4410 & -924 & -2100 & 1260 & -1470 & 2940 & -10584 & 88200 \\ 127008 & -15876 & 4704 & -2646 & 3024 & 0 & -3024 & 2646 & -4704 & 15876 & -127008 \\ -88200 & 10584 & -2940 & 1470 & -1260 & 2100 & 924 & -4410 & 5880 & -17640 & 132300 \\ 43200 & -5040 & 1344 & -630 & 480 & -600 & 1440 & 1914 & -6720 & 15120 & -100800 \\ -14175 & 1620 & -420 & 189 & -135 & 150 & -270 & 945 & 3069 & -11340 & 56700 \\ 2800 & -315 & 80 & -35 & 24 & -25 & 40 & -105 & 560 & 4609 & -25200 \\ -252 & 28 & -7 & 3 & -2 & 2 & -3 & 7 & -28 & 252 & 7381 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \\ \beta_{10} \end{bmatrix}$$

and

$$T_n = -(2520\beta_0 - 252\beta_1 + 56\beta_2 - 21\beta_3 + 12\beta_4 - 10\beta_5 + 12\beta_6 - 21\beta_7 + 56\beta_8 - 252\beta_9 + 2520\beta_{10}) \frac{h^{11}}{27720} y_n^{(11)}$$

I. Eleventh Order Formula:

$$\alpha_0 y_{n+1} = \sum_{i=0}^{10} \alpha_{i+1} y_{n-i} + h \sum_{i=0}^{11} \beta_i y'_{n+1-i}$$

(1.9)

$$27720 \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} = \begin{bmatrix} 83711 & 2520 & -252 & 56 & -21 & 12 & -10 & 12 & -21 & 56 & -252 & 2520 \\ 304920 & -53471 & -5544 & 924 & -308 & 165 & -132 & 154 & -264 & 693 & -3080 & 30492 \\ -762300 & 138600 & -36839 & -9240 & 2310 & -1100 & 825 & -924 & 1540 & -3960 & 17325 & -169400 \\ 1524600 & -207900 & 83160 & -24519 & -13860 & 4950 & -3300 & 3465 & -5544 & 13860 & -59400 & 571725 \\ -2286900 & 277200 & -83160 & 55440 & -14124 & -19800 & 9900 & -9240 & 13860 & -33264 & 138600 & -1306800 \\ 2561328 & -291060 & 77616 & -38808 & 38808 & -4620 & -27720 & 19404 & -25872 & 58212 & -232848 & 2134440 \\ -2134440 & 232848 & -58212 & 25872 & -19404 & 27720 & 4620 & -38808 & 38808 & -77616 & 291060 & -2561328 \\ 1306800 & -138600 & 33264 & -13860 & 9240 & -9900 & 19800 & 14124 & -55440 & 83160 & -277200 & 2286900 \\ -571725 & 59400 & -13860 & 5544 & -3465 & 3300 & -4950 & 13860 & 24519 & -83160 & 207900 & -1524600 \\ 169400 & -17325 & 3960 & -1540 & 924 & -825 & 1100 & -2310 & 9240 & 36839 & -138600 & 762300 \\ -30492 & 3080 & -693 & 264 & -154 & 132 & -165 & 308 & -924 & 5544 & 53471 & -304920 \\ 2520 & -252 & 56 & -21 & 12 & -10 & 12 & -21 & 56 & -252 & 2520 & 83711 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \\ \beta_{10} \\ \beta_{11} \end{bmatrix}$$

and

$$T_n = -(2310\beta_0 - 210\beta_1 + 42\beta_2 - 14\beta_3 + 7\beta_4 - 5\beta_5 + 5\beta_6 - 7\beta_7 + 14\beta_8 - 42\beta_9 + 210\beta_{10} - 2310\beta_{11}) \frac{h^{12}}{27720} y_n^{(12)}$$

APPENDIX II

A. Third Order Formulas

Table 1
α - β Coefficients

r	0	1	2	3
α_0	11	49	257	1548
α_1	18	99	642	4131
α_2	-9	-63	-531	-3672
α_3	2	13	146	1089
β_0	6	30	200	2000
β_1		-18	-280	-4200
β_2			98	2940
β_3				-686

Table 2
Values of the Parameters

r	0	1	2	3
C	0	0.6	0.7	0.7
D	-0.083	-0.014	-0.003	-0.003
$\max\lvert\xi\rvert$, $\xi \neq 1$	0.426	0.515	0.754	0.839
C_{p+1}	-0.1364	-0.1837	-0.3807	-0.8551

B. Fourth Order Formulas

Table 1
α - β Coefficients

r	0	1	2	3	4
α_0	25	91	489	38849	5177
α_1	48	222	472	139934	19024
α_2	-36	-198	-1620	-188922	-26244
α_3	16	82	768	113330	16112
α_4	-3	-15	-131	-25493	-3715
β_0	12	48	300	32000	7500
β_1		-36	-480	-81600	-24000
β_2			192	69360	28800
β_3				-19652	-15360
β_4					3072

Table 2
Values of the Parameters

r	0	1	2	3	4
C	0	0.75	0.8	0.8	0.8
D	-0.667	-0.164	-0.018	-0.007	-0.011
$\max\lvert\xi\rvert$, $\xi\neq 1$	0.561	0.752	0.821	0.858	0.896
C_{p+1}	-0.096	-0.1253	-0.1849	-0.3546	-0.9740

C. Fifth Order Formulas

Table 1
α - β Coefficients

r	0	1	2	3	4	5
α_0	137	512	1877	166528	22157	154637
α_1	300	1395	6090	622875	92700	705200
α_2	-300	-1560	-7860	-925500	-155460	-1288700
α_3	200	980	5180	684500	130760	1179800
α_4	-75	-360	-1815	-252750	-55215	-541175
α_5	12	57	282	37425	9372	99152
β_0	60	240	960	96000	15360	187500
β_1		-180	-1440	-216000	-46080	-750000
β_2			540	162000	51840	1200000
β_3				-40500	-25920	-960000
β_4					4860	384000
β_5						61440

Table 2
Values of Parameters

r	0	1	2	3	4	5
C	0	0.75	0.75	0.75	0.75	0.8
D	-2.327	-0.905	-0.272	-0.067	-0.033	-0.29
$\max\lvert\xi\rvert$, $\xi \neq 1$	0.709	0.749	0.755	0.793	0.832	0.895
C_{p+1}	-0.0730	-0.0898	-0.1156	-0.1596	-0.2507	-0.7455

D. Sixth Order Formulas

Table 1
α - β Coefficients

r	0	1	2	3	4	5	6
α_0	147	284	2094	7725	65539	269927	0.75194
α_1	360	797	7392	31293	307832	1396940	4.20548
α_2	-450	-1050	-11115	-52965	-599970	-3018050	-9.81212
α_3	400	900	9520	48640	621520	3485600	12.22517
α_4	-225	-250	-5070	-26055	-361265	-2270525	-8.57888
α_5	72	159	1584	7875	111864	-711252	3.215
α_6	-10	-22	-217	-1063	-14442	-115290	-0.50269
β_0	60	120	960	3840	37500	187500	1
β_1		-60	-1440	-8640	-120000	-750000	-5.1
β_2			540	6480	144000	1200000	10.8375
β_3				-1620	-76800	-960000	-12.2825
β_4					15360	384000	7.83009
β_5						-61440	-2.66223
β_6							0.37715

Table 2
Values of the Parameters

r	0	1	2	3	4	5	6
C	0	0.5	0.75	0.75	0.8	0.8	0.85
D	-6.071	-3.607	-1.191	-0.409	-0.078	-0.039	-0.051
$\max\lvert\xi\rvert$, $\xi \neq 1$	0.863	0.789	0.760	0.785	0.837	0.896	0.930
C_{p+1}	-0.0583	-0.0654	-0.0843	-0.1071	-0.1569	-0.2521	-0.8605

E. Seventh Order Formulas

Table 1

α - β Coefficients

r	1	2	3	4	5	6	7
α_0	21000	402710	946928	1264853	81200622	1.67631	1.24548
α_1	66717	1469020	3998694	5956076	420194355	9.53216	7.63841
α_2	-104580	-2474801	-7453194	-12184662	-939294720	-23.37143	-20.16032
α_3	111650	2624300	8170015	14201740	1180211550	32.06217	29.69023
α_4	-82600	-1893150	-5803560	-10307185	-903689850	-26.60398	-26.35385
α_5	39375	893060	2674518	4701396	423191097	13.36277	14.10059
α_6	-10892	-245595	-727454	-1252286	-112549080	-3.76441	-4.21113
α_7	1330	29876	87909	149766	13137270	0.45903	0.54155
β_0	$84x10^2$	$168x10^3$	$42x10^4$	$6x10^5$	$42x10^6$	1	1
β_1	-5460	$-2184x10^2$	$-882x10^3$	$-164x10^4$	$-147x10^6$	-4.32	-5.25
β_2		70980	$6174x10^2$	$1764x10^3$	$2058x10^5$	7.776	11.8125
β_3			-144060	$-8232x10^2$	$-14406x10^4$	-7.46496	-14.76563
β_4				144060	$50421x10^3$	4.03108	11.07422
β_5					-7058940	-1.16095	-4.98340
β_6						0.13931	1.24585
β_7							-0.13348

E. Seventh Order Formulas

Table 2
Values of the Parameters

r	1	2	3	4	5	6	7
C	0.65	0.65	0.7	0.7	0.7	0.72	0.75
D	-7.745	-4.142	-1.781	-0.761	-0.315	-0.146	-0.121
$\max\lvert\xi\rvert$, $\xi\neq 1$	0.918	0.819	0.718	0.789	0.805	0.876	0.880
C_{p+1}	-0.0546	-0.0629	-0.0765	-0.0935	-0.1211	-0.1783	-0.3613

F. Eighth Order Formulas

Table 1

α - β Coefficients

r	2	3	4	5	6	7	8
α_0	208185	15837335	1173707	2.06451	1.90132	1.86044	1.61696
α_1	893400	76598820	6215008	12.08742	11.89502	11.60897	10.66413
α_2	-1769838	-165829860	-14564992	-31.05578	-32.67612	-32.02335	-31.03600
α_3	2233280	216404314	20023808	45.96841	51.56311	51.08433	52.08747
α_4	-1996050	-190945650	-17951500	-43.15704	-51.21458	-51.60880	-55.15654
α_5	1250760	117025860	10873632	26.51760	32.85002	33.84335	37.74248
α_6	-514850	-47684420	-4352208	-10.49343	-13.31444	-14.07537	-16.29790
α_7	125088	11515710	1041792	2.45952	3.12358	3.39473	4.05987
α_8	-13605	-1247439	-112233	-0.26207	-0.32527	-0.36341	-0.44654
β_0	84×10^3	672×10^4	525×10^3	1	1	1	1
β_1	-1512×10^2	-17136×10^3	-168×10^4	-4	-4.62	-4.69	-5.36
β_2	68040	145656×10^2	2016×10^3	6.4	8.8935	9.4269	12.5692
β_3		-4126920	-10752×10^2	-5.12	-9.13066	-10.5267	-16.84273
β_4			215040	2.048	5.27295	7.05289	14.10578
β_5				-0.32768	-1.62407	-2.83526	-7.5607
β_6					0.20842	0.63321	2.53283
β_7						-0.06061	-0.48486
β_8							0.04061

F. Eighth Order Formulas

Table 2
Values of the Parameters

r	2	3	4	5	6	7	8
C	0.9	0.85	0.8	0.8	0.77	0.67	0.67
D	-6.913	-3.327	-1.589	-0.572	-0.259	-0.347	-0.292
$\max\lvert\xi\rvert,\ \xi\neq 1$	0.908	0.854	0.817	0.890	0.936	0.823	0.848
C_{p+1}	-0.0562	-0.0631	-0.0785	-0.0998	-0.1268	-0.1369	-0.2026

G. Ninth Order Formulas

Table 1
α - β Coefficients

r	3	4	5	6	7	8	9
α_0	2.49233	2.38020	2.27073	2.16627	1.99009	1.96470	1.91402
α_1	12.99166	13.61294	13.94253	14.02673	13.88969	13.64420	13.34177
α_2	-31.14321	-35.21017	-38.47621	-40.77223	-43.36442	-42.62482	-41.94745
α_3	46.71954	55.14848	63.28275	70.19263	79.63084	78.74159	78.15897
α_4	-49.44600	-58.74204	-69.12855	-79.31378	-94.96159	-94.90653	-95.16604
α_5	38.03174	44.52186	52.48911	61.31624	76.40558	77.46088	78.53287
α_6	-20.70050	-23.86824	-27.80777	-32.55587	-41.54640	-42.82670	-43.90716
α_7	7.50696	8.58943	9.87995	11.46606	14.74203	15.46492	16.02593
α_8	-1.62759	-1.85330	-2.11718	-2.42841	-3.10011	-3.30787	-3.46162
α_9	0.15970	0.18124	0.20612	0.23490	0.29446	0.31903	0.33673
β_0	1	1	1	1	1	1	1
β_1	-2.7	-3.4	-4	-4.5	-5.25	-5.28	-5.4
β_2	2.43	4.335	6.4	8.4375	11.8125	12.1968	12.96
β_3	-0.729	-2.4565	-5.12	-8.4375	-14.76563	-16.09977	-18.144
β_4		0.52201	2.048	4.74609	11.07422	13.28231	16.3296
β_5			-0.32768	-1.42383	-4.9834	-7.01306	-9.79776
β_6				0.17798	1.24585	2.31431	3.9191
β_7					-0.13348	-0.43641	-1.00777
β_8						0.03600	0.16117
β_9							-0.01008

G. Ninth Order Formulas

Table 2
Values of the Parameters

r	3	4	5	6	7	8	9
C	0.9	0.85	0.8	0.75	0.75	0.66	0.6
D	-7.358	-3.720	-1.908	-1.045	-0.445	-0.559	-0.669
$\max\lvert\xi\rvert, \lvert\xi\rvert \neq 1$	0.907	0.881	0.843	0.892	0.919	0.809	0.853
C_{p+1}	-0.0552	-0.0643	-0.0750	-0.0871	-0.1122	-0.1182	-0.1298

H. Tenth Order Formulas

Table 1
α - β Coefficients

r	4	5	6	7	8	9	10
α_0	2.49504	2.44094	2.35486	2.39698	2.25826	2.16251	2.13072
α_1	15.46091	15.64456	15.91264	15.59497	15.88893	15.82665	15.67034
α_2	-43.99996	-46.13535	-49.2588	-47.1692	-51.39958	-53.12941	-52.9723
α_3	77.57873	83.70712	92.82348	88.05197	101.08168	107.97756	108.56297
α_4	-95.82305	-104.87119	-118.9178	-112.89273	-134.22254	-147.34856	-149.48435
α_5	87.19381	95.38026	108.84106	103.93186	125.92003	141.1519	144.4678
α_6	-58.55592	-63.55040	-72.15990	-69.33003	-84.49431	-96.08968	-99.14149
α_7	28.12647	30.30430	34.09694	32.88301	39.96810	45.84452	47.62567
α_8	-9.10910	-9.77631	-10.91500	-10.53770	-12.71736	-14.64359	-15.29600
α_9	1.78241	1.90815	2.12081	2.04686	2.44941	2.82162	2.95944
α_{10}	-0.15927	-0.17020	-0.18859	-0.18203	-0.21610	-0.24849	-0.26138
β_0	1	1	1	1	1	1	1
β_1	-3.68	-4	-4.5	-4.2	-4.96	-5.4	-5.5
β_2	5.0784	6.4	8.43749	7.56	10.7632	12.96	13.6125
β_3	-3.11475	-5.12	-8.43749	-7.56	-13.34637	-18.144	-19.965
β_4	0.71639	2.048	4.74608	4.536	10.34344	16.3296	19.21631
β_5		-0.32768	-1.42382	-1.63296	-5.13034	-9.79776	-12.68277
β_6			0.17798	0.32659	1.59041	3.9191	5.81293
β_7				-0.02799	-0.28173	-1.00777	-1.82692
β_8					0.02183	0.15117	0.37680
β_9						-0.01008	-0.04605
β_{10}							0.00253

H. Tenth Order Formulas

Table 2
Values of the Parameters

r	4	5	6	7	8	9	10
C	0.92	0.8	0.75	0.6	0.62	0.6	0.55
D	-7.442	-5.01	-2.948	-3.572	-1.8	-1.3	-1.2
$\max\|\xi\|,\ \xi\neq 1$	0.934	0.861	0.895	0.884	0.833	0.851	0.890
C_{p+1}	-0.0550	-0.0594	-0.0671	-0.0637	-0.0776	-0.0894	-0.0956

Table 1

α - β Coefficients

r	6	7	8	9	10	11
α_0	2.48725	2.43777	2.39530	2.38615	2.35005	2.32677
α_1	17.81293	17.92604	17.96195	17.87224	17.85686	17.79747
α_2	-59.32633	-61.3217	-62.71853	-62.58565	-63.45946	-63.74501
α_3	122.76072	129.87183	135.37824	135.86498	139.67537	141.45530
α_4	-177.17122	-190.48681	-201.57263	-203.69435	-211.87533	-216.23096
α_5	188.21685	203.88645	217.81506	221.52748	232.48794	238.82776
α_6	-149.91619	-162.40845	-174.17547	-177.99983	-187.91836	-194.02640
α_7	88.86983	95.87160	102.75886	105.30284	111.53332	115.57017
α_8	-38.05524	-40.84183	-43.63909	-44.75489	-47.45853	-49.28591
α_9	11.11267	11.87800	12.64249	12.95729	13.73657	14.28216
α_{10}	-1.97898	-2.10998	-2.23834	-2.29064	-2.42574	-2.52311
α_{11}	0.16221	0.17264	0.18276	0.18669	0.19740	0.20531
β_0	1	1	1	1	1	1
β_1	-4.68	-4.97	-5.2	-5.22	-5.4	-5.5
β_2	9.126	10.5861	11.83	12.1104	13.122	13.75
β_3	-9.49104	-12.52688	-15.379	-16.38941	-18.89568	-20.625
β_4	5.55226	8.89409	12.49544	14.25878	17.85642	20.625
β_5	-1.7323	-3.78888	-6.49763	-8.2701	-11.57096	-14.4375
β_6	0.2252	0.8967	2.11173	3.19777	5.20693	7.21875
β_7		-0.09095	-0.39218	-0.79487	-1.60671	-2.57812
β_8			0.03186	0.11526	0.32536	0.64453
β_9				-0.00743	-0.03904	-0.10742
β_{10}					0.00211	0.01074
β_{11}						-0.00049

I. Eleventh Order Formulas

Table 2
Values of the Parameters

r	6	7	8	9	10	11
C	0.78	0.71	0.65	0.58	0.54	0.5
D	-6.240	-4.487	-3.470	-3.349	-2.953	-2.881
$\max\|\xi\|$, $\xi\neq 1$	0.912	0.886	0.886	0.922	0.937	0.959
C_{p+1}	-0.0559	-0.0601	-0.0640	-0.0651	-0.0689	-0.0716

APPENDIX III

A. Third Order Formulas

Table 1
α - β Coefficients

Polynomial	P_2	Q_2	P_3	Q_3
α_0	185	167	1321	1126
α_1	342	294	2007	1962
α_2	-171	-165	-639	-954
α_3	14	38	-47	118
β_0	96	96	750	750
β_1	0	-24	0	-450
β_2	-54	6	0	270
β_3			-162	-162

Table 2
Values of the Parameters

Polynomial	P_2	Q_2	P_3	Q_3
C	-0.75	-0.25	0.6	-0.6
D	-0.012	-0.06	-0.054	-0.040
$\max\lvert\xi\rvert$, $\xi\neq 1$	0.747	0.477	0.6128	0.5529
C_{p+1}	-0.1054	-0.1587	-0.1726	-0.2558

B. Fourth Order Formulas

Table 1
α - β Coefficients

Polynomial	P_2	Q_2	P_3	Q_3	Q_4
α_0	101	93	1063	185	13897
α_1	200	204	2128	402	29844
α_2	-144	-180	-1884	-342	-23904
α_3	56	84	880	158	8812
α_4	-11	-15	-61	-33	-855
β_0	48	48	500	96	7500
β_1	0	-24	0	-48	-4500
β_2	-12	12	0	24	2700
β_3			256	-12	-1620
β_4					972

Table 2
Values of the Parameters

Polynomial	P_2	Q_2	P_3	Q_3	Q_4
C	-0.5	-0.5	-0.8	-0.5	-0.6
D	-0.557	-0.414	-0.462	-0.472	-0.363
$\max\lvert\xi\rvert$, $\xi \neq 1$	0.527	0.662	0.833	0.578	0.6545
C_{p+1}	-0.0911	-0.1204	-0.0820	-0.1243	-0.1504

C. Fifth Order Formulas

Table 1
α - β Coefficients

Polynomial	P_2	Q_2	P_3	Q_3	P_4	Q_4	Q_5
α_0	551	3137	138458	208	439	2077	4142
α_1	1230	8320	310935	517	964	5150	10225
α_2	-1180	-10220	-343740	-572	-972	-5660	-11120
α_3	740	7160	214580	388	664	3760	7220
α_4	-285	-2515	-53130	-148	-253	-1415	-2530
α_5	46	392	9813	23	36	242	347
β_0	240	1500	60000	96	192	960	1920
β_1	0	-1200	0	-48	0	-480	-960
β_2	-60	960	0	24	0	240	480
β_3			43740	-12	0	-120	-240
β_4					-12	60	120
β_5							-60

Table 2
Values of the Parameters

Polynomial	P_2	Q_2	P_3	Q_3	P_4	Q_4	Q_5
C	-0.5	-0.8	-0.9	-0.5	-0.5	-0.5	-0.5
D	-2.325	-1.077	-1.101	-1.571	-2.250	-1.553	-1.449
$\max\|\xi\|$, $\xi \neq 1$	0.637	0.892	0.949	0.728	0.741	0.6875	0.722
C_{p+1}	-0.0708	-0.0975	-0.0670	-0.0875	-0.0720	-0.0886	-0.0913

D. Sixth Order Formulas

Table 1
α - β Coefficients

Polynomial	Q_2	P_3	Q_3	P_4	Q_5
α_0	3443	18439	5655	1472401	4520
α_1	10156	45576	15655	3619208	12493
α_2	-14810	-59130	-21125	-4572030	-16790
α_3	13280	50000	3760	4192080	14780
α_4	-7105	-25245	-10525	-2334035	-8200
α_5	22208	8424	3306	662376	2615
α_6	-306	-1186	-455	-95198	-378
β_0	1500	7500	2400	$6\text{x}10^5$	1920
β_1	-1200	0	-1200	0	-960
β_2	960	0	600	0	480
β_3		3840	-300	0	-240
β_4				-144060	120
β_5					-60

Table 2
Values of the Parameters

Polynomial	Q_2	P_3	Q_3	P_4	Q_5
C	-0.8	-0.8	-0.5	-0.7	-0.5
D	-2.069	-4.452	-3.808	-4.694	-3.944
$\max\lvert\xi\rvert$, $\xi\neq 1$	0.934	0.868	0.861	0.933	0.839
C_{p+1}	-0.0732	-0.0566	-0.0671	-0.0573	-0.0677

E. Seventh Order Formulas

Table 1
α - β Coefficients

Polynomial	Q_2	P_3
α_0	412310	136400684
α_1	1342600	370394682
α_2	-2287481	-568618092
α_3	2519300	605263295
α_4	-1809150	-430428180
α_5	853160	211815954
α_6	-234675	-59320212
α_7	28556	7293237
β_0	168000	525×10^5
β_1	-142800	0
β_2	121380	0
β_3		28946820

Table 2
Values of the Parameters

Polynomial	Q_2	P_3
C	-0.85	-0.82
D	-4.530	-12.022
$\max\|\xi\|$, $\xi \neq 1$	0.992	0.927
C_{p+1}	-0.0589	-0.0474

LIST OF FIGURES

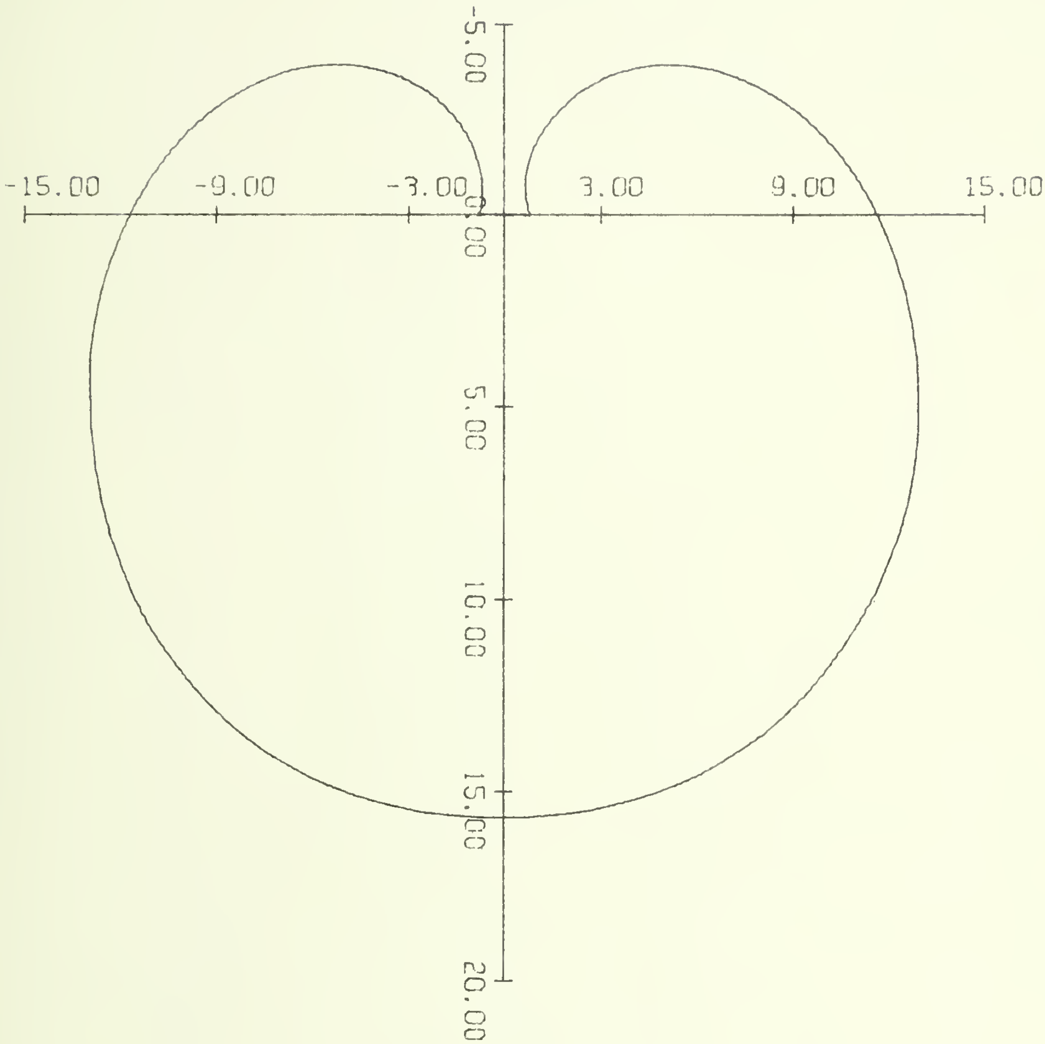

Figure 2

Locus of $\rho(\xi)/\sigma(\xi)$ $\xi = e^{i\Theta}$, $\Theta \varepsilon [0, 2\pi]$

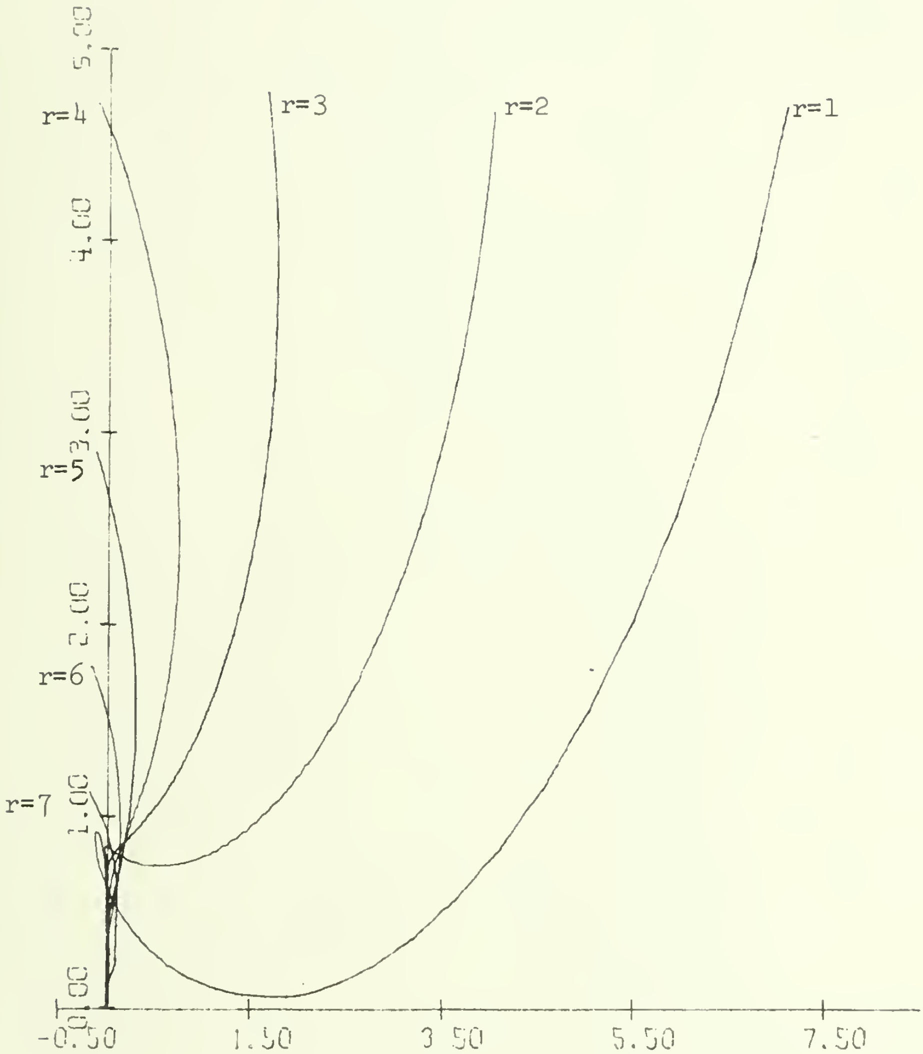

Figure 3

Seventh Order Formulas

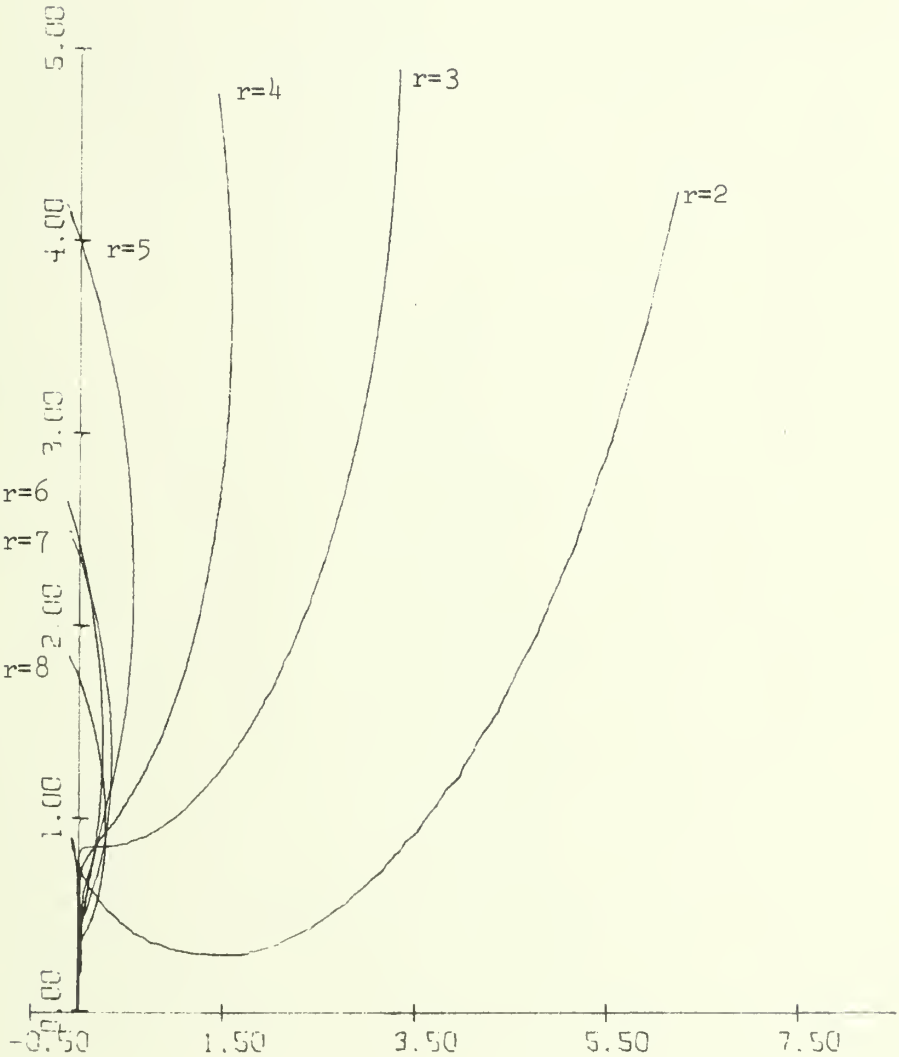

Figure 4

Eighth Order Formulas

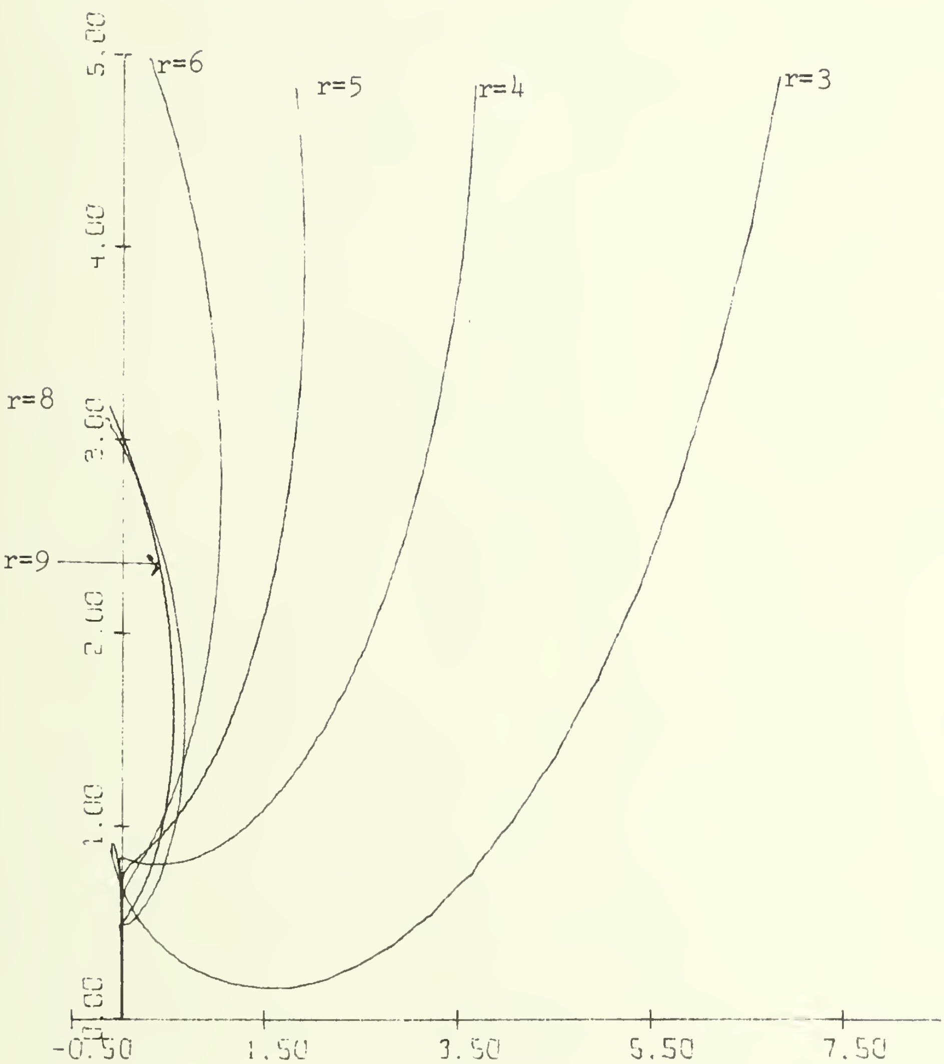

Figure 5

Ninth Order Formulas

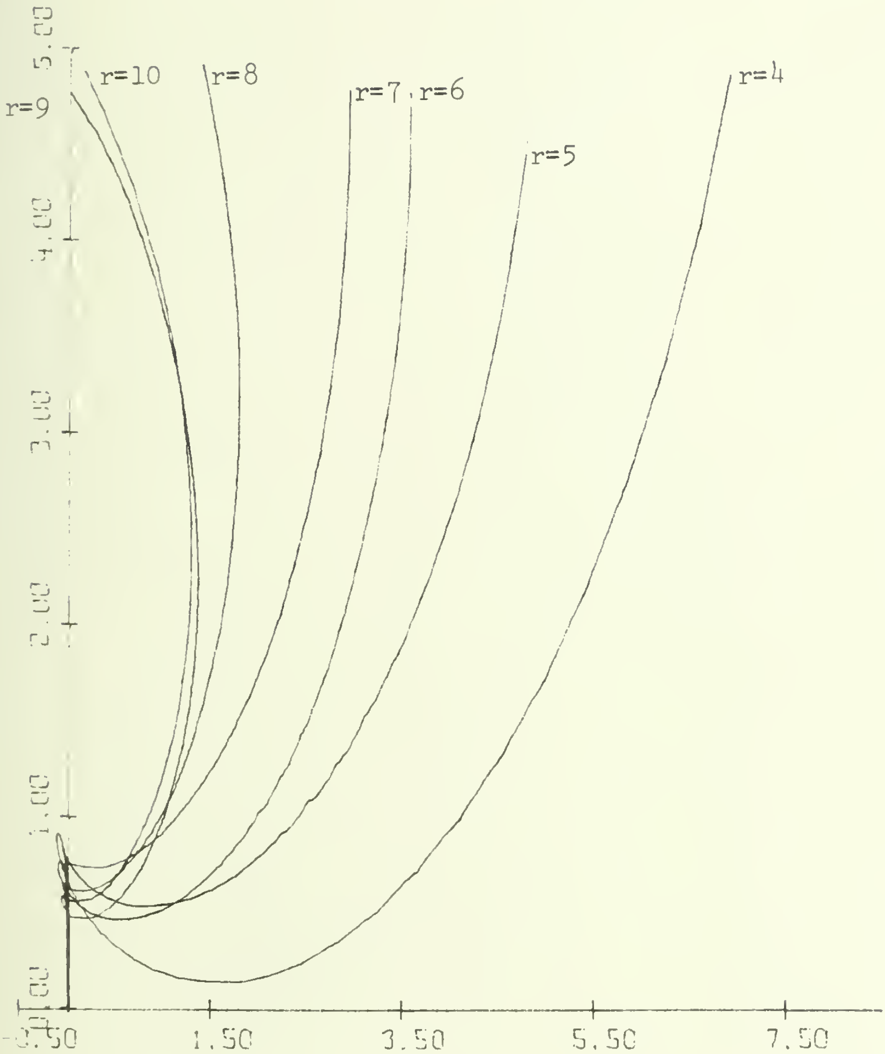

Figure 6

Tenth Order Formulas

Figure 7

Eleventh Order Formulas

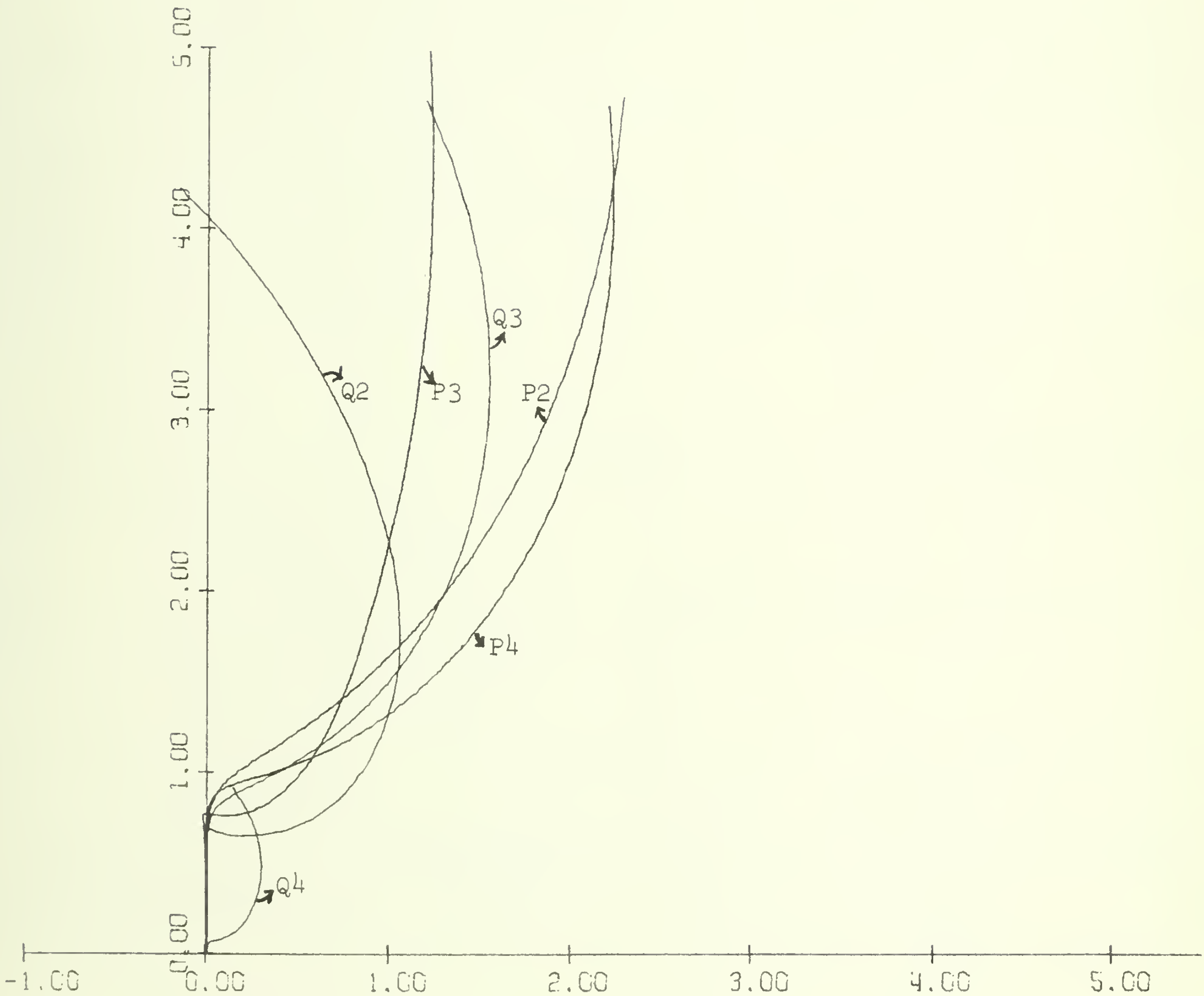

Figure 8

Fifth Order Formula of Class II

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